

## Properties of the Z-Transform

- Linearity:

$$a_1x_1[n] + a_2x_2[n] \longleftrightarrow a_1X_1(z) + a_2X_2(z), \text{RoC} = R_{x_1} \cap R_{x_2}$$

- Time Shifting Property:

$$x[n - n_0] \longleftrightarrow z^{-n_0}X(z),$$

*RoC =  $R_x$  (except possible addition/deletion of  $z = 0$  or  $z = \infty$ )*

- Exponential Weighting:

$$z_0^n x[n] \longleftrightarrow X(z_0^{-1}z), \text{RoC} = |z_0|R_x$$

– The poles of the Z-transform are scaled by  $|z_0|$

- Linear Weighting

$$nx(n) \longleftrightarrow -z \frac{dX(z)}{dz},$$

$RoC = R_x$  (except possible addition/deletion of  $z = 0$  or  $z = \infty$ )

- Time Reversal

$$x[-n] \longleftrightarrow X(z^{-1}), RoC = \frac{1}{R_x}$$

- Convolution

$$x[n] * y[n] \longleftrightarrow X(z)Y(z), RoC = R_x \cap R_y$$

- Multiplication

$$x[n]w[n] \longleftrightarrow \frac{1}{2\pi j} \int X(v)w\left(\frac{z}{v}\right)v^{-1}dv$$

## Inverse Z-Transform Examples

- Using long division: Causal sequence

$$\frac{1}{1 - az^{-1}}, \text{RoC} = |z| > |a| = 1 + az^{-1} + az^{-2} + az^{-3} + \dots$$

$$\text{IZT}(1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \dots) = a^n u[n]$$

- Using long division: Noncausal sequence

$$\frac{1}{1 - az^{-1}}, \text{RoC} = |z| < |a|$$

Here the IZT is computed as follows:

$$\text{IZT}\left(\frac{1}{1 - az^{-1}}\right) = \text{IZT}\left(\frac{z}{-a + z}\right)$$

This results in:

$$\text{IZT}(-a^{-1}z + a^{-2}z^2 + a^{-3}z^3 + \dots) = -a^n u[-n - 1]$$

- Inverse  $Z$ -transform - using Power series expansion

$$X(z) = \log(1 + az^{-1}), |z| > |a|$$

Using the Power Series expansion for  $\log(1 + x)$ ,  $|x| < 1$ , we have

$$X(z) = \sum_1^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

The IZT is given by

$$\begin{aligned} x[n] &= \frac{(-1)^{n+1} a^n}{n}, n \geq 1 \\ &= 0, n \leq 0 \end{aligned}$$

- Inverse  $Z$ -transform - Inspection method

$$a^n u[n] \longleftrightarrow \frac{1}{1 - az^{-1}}, |z| > |a|$$
$$\text{Given } X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, |z| > \left|\frac{1}{2}\right|$$
$$\implies x[n] = \left(\frac{1}{2}\right)^n u[n]$$

- Inverse  $Z$ -transform - Partial fraction method
  - Example 1: All-Pole system

$$X(z) = \frac{1}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{6}z^{-1}\right)}, |z| > \frac{1}{3}$$

Using partial fraction method, we have:

$$X(z) = \frac{A_1}{1 - \frac{1}{3}z^{-1}} + \frac{A_2}{1 - \frac{1}{6}z^{-1}},$$

$$|z| > \frac{1}{3}$$

$$A_1 = \left(1 - \frac{1}{3}z^{-1}\right)X(z)\Big|_{z=\frac{1}{3}}$$

$$A_2 = \left(1 - \frac{1}{6}z^{-1}\right)X(z)\Big|_{z=\frac{1}{6}}$$

$$A_1 = 2$$

$$A_2 = -1$$

$$x(n) = 2\left(\frac{1}{3}\right)^n u[n] - 1\left(\frac{1}{6}\right)^n u[n]$$

– Example 2: Pole-Zero system

$$\begin{aligned} X(z) &= \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, |z| > 1 \\ &= \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \\ &= 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \\ &= 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}} \\ x[n] &= 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n] \end{aligned}$$

– Example 3: Finite length sequences

$$\begin{aligned} X(z) &= z^2 \left(1 - \frac{1}{2}z^{-1}\right) (1 + z^{-1}) (1 - z^{-1}) \\ &= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1} \\ &= \delta[n + 2] - \frac{1}{2}\delta[n + 1] - \delta[n] + \frac{1}{2}\delta[n - 1] \end{aligned}$$

## Inverse Z-Transform Problem

1. Given  $X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{z-3}$ , determine all the possible sequences for  $x[n]$ .

Hint: Remember that the RoC must be a continuous region